Non-pointlike particles in harmonic oscillators

1997 J. Phys. A: Math. Gen. 30 2093

(http://iopscience.iop.org/0305-4470/30/6/030)

J. Phys. A: Math. Gen. **30** (1997) 2093–2101. Printed in the UK PII: S0305-4470(97)78859-0

Non-pointlike particles in harmonic oscillators

Achim Kempf†‡

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, UK

Received 22 October 1996, in final form 30 December 1996

Abstract. Quantum mechanics usually describes particles as being pointlike in the sense that, in principle, the uncertainty, Δx, can be made arbitrarily small. Studies on string theory and quantum gravity motivate correction terms to the uncertainty relations which induce a finite lower bound Δx_0 to spatial localization. This structure is implemented into quantum mechanics through small correction terms to the canonical commutation relations. We calculate the perturbations to the energy levels of a particle which is non-pointlike in this sense in isotropic harmonic oscillators, where we find a characteristic splitting of the usually degenerate energy levels. Possible applications are outlined.

1. Introduction

The existence of a fundamental short distance cut-off has long been conjectured, for well known conceptual and technical reasons. In this context, recent studies on string theory and quantum gravity yielded correction terms to the uncertainty relations which imply a finite minimal uncertainty Δx_0, e.g. at the Planck scale. A minimal uncertainty, Δx_0, may therefore be viewed both as a fuzziness of space-time, or as a fundamental non-pointlikeness of the elementary 'particles' as strings. For examples see [1–4], more recently [5], and recent reviews are [6, 7]. Technically, we will implement the string/quantum gravity uncertainty principle through small correction terms to the canonical commutation relations [8, 9].

With the quantum mechanical description of a new short distance structure at hand one can then ask whether, adjusting the scale appropriately, it may provide a new model for an effective description not only of strings but also of non-pointlike particles such as quasi-particles and various collective excitations in solids, or composite particles such as nucleons and nuclei. For example, nucleons in nuclear potentials or nuclei in molecular potentials have simple effective quantum mechanical descriptions which normally implicitly assume them to have a pointlike charge distribution. It should be worth investigating whether the introduction of a Δx_0 could provide an effective description which, while still technically being relatively simple, could within some limits correctly account for effects caused by the non-pointlike nature, i.e. by the finite spread of the charge distribution, of these particles.

As an example system we study the problem of a particle which is non-pointlike in this sense in a d-dimensional isotropic harmonic oscillator. The special case of the one-dimensional oscillator was solved in [10]. The application of different methods now allows us to perturbatively calculate the spectra for the general d-dimensional case. Our result, a

† Research Fellow of Corpus Christi College in the University of Cambridge
‡ E-mail address: a.kempf@amtp.cam.ac.uk

0305-4470/97/062093+09$19.50

characteristic splitting of the usually degenerate energy levels, has only one free parameter, which is the spatial extend of the oscillating particle. The ansatz is therefore disprovable for any actual harmonic oscillator system in which the effects of the finite size of the oscillating particle on the spectrum are measurable.

As an important side-result we show that it is possible to introduce a finite minimal uncertainty, Δx_0, without breaking conventional rotation and translation invariance, for which we give the general condition.

A considerable amount of recent literature exists on various deformations of the harmonic oscillator, most of which specifically investigates the possibilities of quantum group symmetry, see e.g. [11–16], which is related to noncommutative geometry [17]. We remark that our ansatz also originated from studies on quantum group symmetries [18].

2. Heisenberg algebra

We begin with the simple one-dimensional case by considering the associative Heisenberg algebra generated by $\boldsymbol{x}$ and $\boldsymbol{p}$ with commutation relations

$$[\boldsymbol{x}, \boldsymbol{p}] = \mathrm{i}\hbar(1 + \beta \boldsymbol{p}^2) \tag{1}$$

for some small $\beta > 0$. The commutation relation yields the uncertainty relation:

$$\Delta x \Delta p \geqslant \hbar/2(1 + \beta(\Delta p)^2 + \beta\langle \boldsymbol{p}^2\rangle). \tag{2}$$

This uncertainty relation is of the type considered in [6, 7], and as is readily checked it implies a minimal uncertainty in positions $\Delta x_0 = \hbar\sqrt{\beta}$.

In d dimensions we consider

$$[\boldsymbol{x}_i, \boldsymbol{p}_j] = \mathrm{i}\hbar\Theta_{\mathrm{i}j}(\boldsymbol{p}) \tag{3}$$

with appropriate choices of symmetric Θ that induce a minimal uncertainty $\Delta x_0 > 0$. We will leave momentum space 'classical', i.e. $[\boldsymbol{p}_i, \boldsymbol{p}_j] = 0$. The Jacobi identity and the requirements $\boldsymbol{x}_i = \boldsymbol{x}_i^\dagger$, $\boldsymbol{p}_i = \boldsymbol{p}_i^\dagger$ then uniquely determine the commutation relations for the, now in the generic case, noncommutative position operators:

$$[\boldsymbol{x}_i, \boldsymbol{x}_j] = \mathrm{i}\hbar\{\boldsymbol{x}_a, \Theta^{-1}_{ar}\Theta_{s[i}\Theta_{j]r,s}\}. \tag{4}$$

For simplicity we adopted a geometric notation with $f_{,s}$ standing for $\partial_{p_s} f$ and where repeated indices are summed over. The associative Heisenberg algebra $\mathcal{A}$ finds a Hilbert space representation, e.g. on momentum space through [19]:

$$\boldsymbol{p}_i.\psi(p) = p_i\psi(p) \tag{5}$$

$$\boldsymbol{x}_i.\psi(p) = \mathrm{i}\hbar(\tfrac{1}{2}\Theta_{ai,a} + \Theta_{ai}\partial_{p_a})\psi(p). \tag{6}$$

If we require rotational isotropy and assume the minimal uncertainty Δx_0 to be small, the lowest-order correction terms to the canonical commutation relations read

$$\Theta_{ij}(p) = \delta_{ij} + \beta\delta_{ij}\boldsymbol{p}^2 + \beta'\boldsymbol{p}_i\boldsymbol{p}_j \tag{7}$$

where $\boldsymbol{p}^2 := \sum_{i=1}^d \boldsymbol{p}_i\boldsymbol{p}_i$, and where $\beta, \beta' > 0$ are assumed to be small and of the first order. The same mechanism as in the one-dimensional case yields from the corresponding uncertainty relations $\Delta x_i \Delta p_i \geqslant |\langle[\boldsymbol{x}_i, \boldsymbol{p}_j]\rangle|/2$ an isotropic (i.e. $\Delta x_{0i} = \Delta x_{0j}, \forall i, j$) minimal uncertainty Δx_0 (dropping the index i):

$$\Delta x_0 = \hbar\sqrt{\beta d + \beta'}. \tag{8}$$

The commutation relations in the Heisenberg algebra $\mathcal{A}$ then read $[\boldsymbol{p}_i, \boldsymbol{p}_j] = 0$, and:

$$[\boldsymbol{x}_i, \boldsymbol{p}_j] = \mathrm{i}\bar{h}(\delta_{ij} + \beta\delta_{ij}\boldsymbol{p}^2 + \beta'\boldsymbol{p}_i\boldsymbol{p}_j) + \mathcal{O}^2(\beta, \beta') \tag{9}$$

$$[\boldsymbol{x}_i, \boldsymbol{x}_j] = \mathrm{i}\bar{h}(\beta'/2 - \beta)(\{\boldsymbol{x}_i, \boldsymbol{p}_j\} - \{\boldsymbol{x}_j, \boldsymbol{p}_i\}) + \mathcal{O}^2(\beta, \beta'). \tag{10}$$

In the momentum representation, where

$$\boldsymbol{x}_i.\psi(p) = \mathrm{i}\bar{h}\left[\left(\beta + \beta'\frac{d+1}{2}\right)p_i + (\delta_{ia} + \beta\delta_{ia}p^2 + \beta' p_i p_a)\partial_{p_a}\right]\psi(p) \tag{11}$$

we have to this order:

$$[\boldsymbol{x}_i, \boldsymbol{x}_j].\psi(p) = \bar{h}^2(\beta' - 2\beta)(p_i\partial_{p_j} - p_j\partial_{p_i})\psi(p) + \mathcal{O}^2(\beta, \beta'). \tag{12}$$

In the case $\beta = \beta'/2$ the $\boldsymbol{x}_i$ are commutative, which may therefore be considered a preferred choice of parameters. The framework is then translation invariant, i.e. $\boldsymbol{p}_i \to \boldsymbol{p}_i$, $\boldsymbol{x}_i \to \boldsymbol{x}_i + \alpha_i$ defines an algebra homomorphism of $\mathcal{A}$. More generally, this feature holds for any Θ that obeys (from equation (4)):

$$\Theta_{ia}\partial_{p_i}\Theta_{bc} = \Theta_{ib}\partial_{p_i}\Theta_{ac}. \tag{13}$$

In the rotation symmetric case

$$\Theta_{ij} := \delta_{ij}f(p^2) + g(p^2)p_i p_j \tag{14}$$

this condition then reads

$$g = 2ff'(f - 2p^2 f')^{-1} \tag{15}$$

where we may choose, e.g. $f := \mathrm{e}^{\beta p^2}$ to recover to first-order equation (7) with $\beta' = 2\beta$. Having established that translation and rotation invariance are preserved, let us briefly discuss how far the approach does differ from the usual quantum mechanical treatment. Corrections to the commutation relations indeed necessarily induce new physical features, such as here the appearance of a minimal uncertainty in positions. These features could not alternatively be described by keeping the ordinary commutation relations and instead adding corrections to Hamiltonians. This is because for the full set of predictions, such as spectra, transition probabilities and expectation values to match, systems must be related by a unitary transformation. However, unitary transformations are commutation relations preserving: For any commutation relation $g(\boldsymbol{x}, \boldsymbol{p}) = 0$ with g being a polynomial in $\boldsymbol{x}$ and $\boldsymbol{p}$, and all unitary transformations $U : \boldsymbol{x} \to \boldsymbol{x}' = U\boldsymbol{x}U^\dagger, \boldsymbol{p} \to \boldsymbol{p}' = U\boldsymbol{p}U^\dagger$ there holds

$$g(\boldsymbol{x}, \boldsymbol{p}) = 0 \;\Rightarrow\; 0 = Ug(\boldsymbol{x}, \boldsymbol{p})U^\dagger = g(U\boldsymbol{x}U^\dagger, U\boldsymbol{p}U^\dagger) = g(\boldsymbol{x}', \boldsymbol{p}'). \tag{16}$$

In our case of generalized commutation relations which imply a $\Delta x_0 > 0$, an important technical consequence is that the generalized commutation relations no longer find a spectral representation of the $\boldsymbol{x}_i$. To see this, note that generally $Q|\lambda\rangle = \lambda|\lambda\rangle \Rightarrow (\Delta Q)_{|\lambda\rangle} = 0$. For the details of the functional analysis see [8]. We remark that in this context the concept of quasi-position representation has been introduced in [10]. The quasi-position wavefunction of a state $|\psi\rangle$ is defined through $\psi(\xi) := \langle\phi_\xi^{ml}|\psi\rangle$ where the $|\phi_\xi^{ml}\rangle$ are states of now maximal spatial localization around positions ξ. The states of maximal spatial localization $|\phi_\xi^{ml}\rangle$ obeying

$$\langle\phi_\xi^{ml}|\boldsymbol{x}|\phi_\xi^{ml}\rangle = \xi \qquad \Delta x_{|\phi_\xi^{ml}\rangle} = \Delta x_0 \tag{17}$$

have been studied in [10, 20], and they reduce of course to the position eigenstates for $\Delta x_0 \to 0$. In [21] the maximal localization states have been calculated for commutation relations which also include a $\Delta p_0 > 0$.

Finally, we remark that the generalized commutation relations are not Galilean invariant, i.e. there formally exists a preferred rest frame. This is, however, an artefact of our choice to only generalize the spatial short distance structure while leaving the time coordinate unchanged. In four dimensions euclidean rotation and translation invariance turns into Poincaré invariance upon Wick rotation. For studies on euclidean quantum field theory with the generalized short distance structure see [9, 20], and recently [22].

3. The harmonic oscillator

A low energy approximation for most kinds of oscillations is a d-dimensional harmonic oscillator, which we, here for simplicity, choose isotropic:

$$H := \sum_{i=1}^{d} \left(\frac{\boldsymbol{p}_i^2}{2m} + \frac{m\omega^2 \boldsymbol{x}_i^2}{2} \right). \tag{18}$$

To see that for $\Delta x_0 > 0$, H still has a unique diagonalization, assume the commutation relations and H represented on some dense domain D in the Hilbert space (below, our choice of D will be the S_∞ functions on momentum space). Since $\boldsymbol{p}^2$ is positive and $\boldsymbol{x}^2$ is positive *definite* ($\langle \boldsymbol{x}^2 \rangle \geqslant (\Delta x_0)^2$), also H is positive definite on D. Therefore, H has exactly one self-adjoint extension in its form domain (see, e.g. [9] and references therein), with the form domain of H being the common domain of $\boldsymbol{x}$ and $\boldsymbol{p}$, i.e. the physical space of states.

In Hilbert space representations of the generalized commutation relations the usual perturbative techniques for the calculation of eigenvalues of Hermitian operators, such as Hamiltonians, are therefore still applicable. The Hamiltonian acts on momentum space as

$$H.\psi(p) = \left[\frac{p^2}{2m} - \frac{m\omega^2 \overline{h}^2}{2} \sum_{i=1}^{d} \left(\left(\beta + \beta' \frac{d+1}{2} \right) p_i + \partial_{p_i} + \beta p^2 \partial_{p_i} + \beta' p_i p_a \partial_{p_a} \right)^2 \right] \psi(p) \tag{19}$$

which is, to first order in β, β':

$$H.\psi(p) = \left[\frac{p^2}{2m} - \frac{m\omega^2 \overline{h}^2}{2} (\beta d + \beta'(d + d^2)/2 + (4\beta + (2 + 2d)\beta') p_i \partial_{p_i} + 2\beta p^2 \partial^2 + 2\beta' p_i p_a \partial_{p_i} \partial_{p_a} + \partial^2) \right] \psi(p). \tag{20}$$

Since we are dealing with harmonic oscillators it is convenient to further transform into a Fock representation where $|\psi\rangle = \psi(a^\dagger)|0\rangle$. The multiplication and differentiation operators p_i and ∂_{p_j} can be represented through

$$p_j.|\psi\rangle = \mathrm{i}(m\omega\overline{h}/2)^{1/2}(a_j^\dagger - a_j)|\psi\rangle \tag{21}$$

$$\partial_{p_j}.|\psi\rangle = -\mathrm{i}(2m\omega\overline{h})^{-1/2}(a_j^\dagger + a_j)|\psi\rangle \tag{22}$$

where $a_i a_j^\dagger - a_j^\dagger a_i = \delta_{ij}$, so that

$$p_i \partial_i = -\frac{d}{2} + \text{NN-terms} \tag{23}$$

$$p^2 \partial^2 = -N^2 - N(d+1) - \frac{d(d+2)}{4} + \frac{1}{2} \sum_{i,j=1}^{d} a_i^2 a_j^{\dagger 2} + \text{NN-terms} \tag{24}$$

$$p_i p_j \partial_i \partial_j = N + \frac{d(d+4)}{4} - \frac{1}{2}\sum_{i,j=1}^{d} a_i^2 a_j^{\dagger 2} + \text{NN-terms} \tag{25}$$

where $N_i := a_i^\dagger a_i$, $N := \sum_{i=1}^d N_i$, and where NN-terms are terms that contain nonequal numbers of raising and lowering operators. Substituting equations (23)–(25) into equation (20) yields for the action of H on Fock space:

$$H|\psi\rangle = \left[\hbar\omega\left(N + \frac{d}{2}\right) + m\omega^2\hbar^2(\beta N^2 + (\beta(d+1) - \beta')N + (\beta d(d+4) - 3\beta' d)/4)\right.$$
$$\left. -m\omega^2\hbar^2\frac{\beta - \beta'}{2}\sum_{i,j=1}^{d} a_i^2 a_j^{\dagger 2} + \text{NN-terms}\right]|\psi\rangle. \tag{26}$$

The natural length scale of the harmonic oscillator is the inverse length in the exponent of the Hermite functions: $(\hbar/m\omega)^{1/2}$. Let us replace the parameters β and β' by more intuitive dimensionless parameters k, k' which measure the minimal uncertainty length scales associated with β and β' in units of the length scale of the oscillator (see equation (8)): $k := \hbar\sqrt{\beta}/(\hbar/m\omega)^{1/2}$, $k' := \hbar\sqrt{\beta'}/(\hbar/m\omega)^{1/2}$, i.e. we have $\beta = k^2/m\omega\hbar$, $\beta' = k'^2/m\omega\hbar$ and thus, from equation (8),

$$\Delta x_0 = \sqrt{k^2 d + k'^2}\sqrt{\frac{\hbar}{m\omega}} \tag{27}$$

so that

$$H|\psi\rangle = \hbar\omega\left[N + \frac{d}{2} + k^2 N^2 + \left(k^2(d+1) - k'^2\right)N + \frac{k^2 d(d+4) - 3k'^2 d}{4}\right.$$
$$\left. -\frac{k^2 - k'^2}{2}\sum_{i,j=1}^{d} a_i^2 a_j^{\dagger 2} + \text{NN-terms}\right]|\psi\rangle. \tag{28}$$

4. First-order corrections to the spectra

We read from equation (28) that H consists of a diagonal part with degenerate eigenvalues (in more than one dimension), and a nondiagonal term $\sum_{i,j=1}^d a_i^2 a_j^{\dagger 2}$ proportional to $(k^2 - k'^2)$. As a new effect, this nondiagonal term can lead to a splitting of the normally $g(n,d)$-fold degenerate eigenvalues E_n of the d-dimensional isotropic harmonic oscillator. We recall the degeneracy function: $g(n,d) = \frac{(n+d-1)!}{n!(d-1)!}$. As is well known, in the calculation of the eigenvalues of $A := B + C$, for A, B, C Hermitian, the first-order perturbative corrections to degenerate eigenvalues of B are the eigenvalues of the perturbing matrix C when restricted to the corresponding eigenspaces.

Thus, here the $g(n,d)$-fold degenerate energy levels E_n split into levels E'_{n_r}:

$$E'_{n_r}(k,k') = \hbar\omega\left(n + \frac{d}{2} + k^2 n^2 + k^2(d+1)n - k'^2 n + \frac{k^2 d(d+4) - 3k'^2 d}{4} - \frac{k^2 - k'^2}{2}\right.$$
$$\left. \times r\text{th eigenvalue}\left[\left(\sum_{i,j=1}^{d} a_i^2 a_j^{\dagger 2}\right)\Big|_{\mathcal{H}_n}\right]\right) \tag{29}$$

where $r = 1, 2, \ldots, g(n,d)$ and where the eigenspaces $\mathcal{H}_n$ of the diagonal part of the Hamiltonian are

$$\mathcal{H}_n := \text{span}\left\{(r_1! \cdot \ldots \cdot r_d!)^{-1/2} a_1^{\dagger r_1} \cdot \ldots \cdot a_d^{\dagger r_d}|0\rangle \,\middle|\, \sum_{i=1}^{d} r_i = n\right\}. \tag{30}$$

The matrix elements of NN-terms vanish in $\mathcal{H}_n$, i.e. for $m = \sum_{i=1}^{d} r_i = \sum_{i=1}^{d} s_i$ there holds $\langle 0|a_1^{s_1} \cdot \ldots \cdot a_d^{s_d}(\text{NN-terms})a_1^{\dagger r_1} \ldots a_d^{\dagger r_d}|0\rangle = 0$, so that the NN-terms of equation (28) do not contribute in equation (29).

For the calculation of the eigenvalues of $\sum_{i,j=1}^{d} a_i^2 a_j^{\dagger 2}$ in $\mathcal{H}_n$ we can choose the *ON*-basis given in equation (30) to obtain the matrix elements ($n = \sum_{i=1}^{d} r_i = \sum_{i=1}^{d} s_i$):

$$\langle r_1, \ldots, r_d| \sum_{i,j}^{d} a_i^2 a_j^{\dagger 2}|s_1, \ldots, s_d\rangle = \sum_{i,j=1}^{d} \sqrt{(r_i+1)(r_i+2)(s_j+1)(s_j+2)} \times \delta_{r_1,s_1} \ldots \delta_{r_i+2,s_i} \ldots \delta_{r_j,s_j+2} \ldots \delta_{r_d,s_d}. \tag{31}$$

We begin with the one-dimensional case. For this case the momentum space Schrödinger equation proved to be exactly solvable in terms of hypergeometric functions, yielding to first order in β, from equations (53), (56), (69) of [10]:

$$E'_n = \hbar\omega(n + \tfrac{1}{2}) + m\omega^2\hbar^2\beta(n^2/2 + n/2 + \tfrac{1}{4}). \tag{32}$$

Indeed, we recover this result from equations (26) and (29) as the special case $d = 1$ (note that $a^2 a^{\dagger 2} = N^2 + 3N + 2$ and that $\beta + \beta'$, corresponds to β in [10]).

For $d = 2$, a straightforward calculation now yields the $g(n, 2) = n + 1$ eigenvalues of $\sum_{i,j=1}^{2} a_i^2 a_j^{\dagger 2}$ in $\mathcal{H}_n$. If n is odd, these can be put into the form $4s(n + 2 - s)$ for $s = 1, \ldots, (n + 1)/2$ with all eigenvalues two-fold degenerate. For n even, s runs $s = 1, \ldots, (n + 2)/2$ with the last eigenvalue nondegenerate. Using equation (29) we therefore obtain the energy levels for $d = 2$ (with the degeneracies given within brackets):

$$E'_{n_r}(k, k') = \hbar\omega\Big[n + 1 + k^2(n^2 + 3n + 3) - k'^2(n + \tfrac{3}{2}) - (k^2 - k'^2)\begin{cases} 2 \cdot 1(n + 2 - 1) & (2\times) \\ 2 \cdot 2(n + 2 - 2) & (2\times) \\ 2 \cdot 3(n + 2 - 3) & (2\times) \\ \vdots & \\ (n + 1)(n + 3)/2 & (2\times) \text{ for } n \text{ odd} \\ (n + 2)^2/2 & (1\times) \text{ for } n \text{ even}. \end{cases} \tag{33}$$

Of special interest is the three-dimensional oscillator. To illustrate the calculation, consider e.g. the splitting of the second excited energy level E_2. We may choose as an ON-basis of $\mathcal{H}_2$ (see equation (30)): $e_1 := |2, 0, 0\rangle$, $e_2 := |0, 2, 0\rangle$, $e_3 := |0, 0, 2\rangle$, $e_4 := |0, 1, 1\rangle$, $e_5 := |1, 0, 1\rangle$, $e_6 := |1, 1, 0\rangle$, in which:

$$\left(\sum_{i,j=1}^{d} a_i^2 a_j^{\dagger 2}\right)\Big|_{\mathcal{H}_2} = \begin{pmatrix} 16 & 2 & 2 & 0 & 0 & 0 \\ 2 & 16 & 2 & 0 & 0 & 0 \\ 2 & 2 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 14 & 0 & 0 \\ 0 & 0 & 0 & 0 & 14 & 0 \\ 0 & 0 & 0 & 0 & 0 & 14 \end{pmatrix}. \tag{34}$$

The eigenvalues are: 14, 14, 14, 14, 14, 20. Thus, the ordinarily six-fold degenerate second-excited level, E_2, splits into two energy levels, one of which is five-fold degenerate and one nondegenerate (see equation(37)). The calculation of the first few excited states shows the systematics in the splitting of the levels:

$$E'_0 = \hbar\omega\left[\frac{3}{2} + \frac{21k^2 - 9k'^2}{4} - (k^2 - k'^2) \cdot 3\ (1\times)\right] \tag{35}$$

$$E'_{1_r} = \hbar\omega\left[\frac{5}{2} + \frac{41k^2 - 13k'^2}{4} - (k^2 - k'^2)\cdot 5\ (3\times)\right] \tag{36}$$

$$E'_{2_r} = \hbar\omega\left[\frac{7}{2} + \frac{69k^2 - 17k'^2}{4} - (k^2 - k'^2)\cdot \begin{Bmatrix} 7\ (5\times) \\ 10\ (1\times) \end{Bmatrix}\right] \tag{37}$$

$$E'_{3_r} = \hbar\omega\left[\frac{9}{2} + \frac{105k^2 - 21k'^2}{4} - (k^2 - k'^2)\cdot \begin{Bmatrix} 9\ (7\times) \\ 14\ (3\times) \end{Bmatrix}\right] \tag{38}$$

$$E'_{4_r} = \hbar\omega\left[\frac{11}{2} + \frac{149k^2 - 25k'^2}{4} - (k^2 - k'^2)\cdot \begin{Bmatrix} 11\ (9\times) \\ 18\ (5\times) \\ 21\ (1\times) \end{Bmatrix}\right] \tag{39}$$

$$E'_{5_r} = \hbar\omega\left[\frac{13}{2} + \frac{201k^2 - 29k'^2}{4} - (k^2 - k'^2)\cdot \begin{Bmatrix} 13\ (11\times) \\ 22\ (7\times) \\ 27\ (3\times) \end{Bmatrix}\right]. \tag{40}$$

5. The translation invariant case

The splitting of the energy levels for the case $k'^2 = 2k^2$ is interesting since the framework is then translation invariant and the x_i commute (equation (12)). (Note that there would be no splitting for $k'^2 = k^2$.)

$$E'_0 = \tfrac{3}{2}\hbar\omega + \tfrac{3}{4}(\Delta x_0)^2 m\omega^2\ (1\times) \tag{41}$$

$$E'_{1_r} = \tfrac{5}{2}\hbar\omega + \tfrac{3}{4}(\Delta x_0)^2 m\omega^2\ (3\times) \tag{42}$$

$$E'_{2_r} = \tfrac{7}{2}\hbar\omega + (\Delta x_0)^2 m\omega^2 \cdot \begin{Bmatrix} \frac{63}{20}\ (5\times) \\ \frac{15}{4}\ (1\times) \end{Bmatrix} \tag{43}$$

$$E'_{3_r} = \tfrac{9}{2}\hbar\omega + (\Delta x_0)^2 m\omega^2 \cdot \begin{Bmatrix} \frac{99}{20}\ (7\times) \\ \frac{119}{20}\ (3\times) \end{Bmatrix} \tag{44}$$

$$E'_{4_r} = \tfrac{11}{2}\hbar\omega + (\Delta x_0)^2 m\omega^2 \cdot \begin{Bmatrix} \frac{143}{20}\ (9\times) \\ \frac{171}{20}\ (5\times) \\ \frac{183}{20}\ (1\times) \end{Bmatrix} \tag{45}$$

$$E'_{5_r} = \tfrac{13}{2}\hbar\omega + (\Delta x_0)^2 m\omega^2 \cdot \begin{Bmatrix} \frac{195}{20}\ (11\times) \\ \frac{231}{20}\ (7\times) \\ \frac{251}{20}\ (3\times) \end{Bmatrix}. \tag{46}$$

Equations (41)–(46) give the first few levels for this case and $d = 3$, expressed in terms of the, then only, free parameter Δx_0. The removal of the degeneracy implies that $\Delta x_0 > 0$ breaks the accidental dynamical $SU(n)$ symmetry of the harmonical oscillator. On the other hand, due to the conservation of the $SO(n)$ symmetry, m-degeneracy is not removed. This allows us to read the angular momenta of the perturbed levels from their remaining degeneracy. We include the angular momentum quantum numbers on the graph of the spectrum in figure 1.

We observe that, as an overall feature of the model, the effect of the non-pointlikeness generally perturbs the energy levels upwards. These perturbations therefore characteristically go in the opposite direction to the in oscillators normally to be expected energy level lowering effect of the eventual anharmonic flattening of the potential.

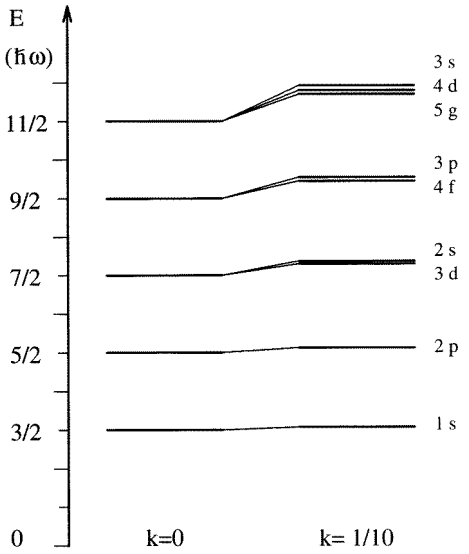

Figure 1. Splitting of the energy levels of the three-dimensional isotropic harmonic oscillator for a non-pointlike particle with relative size $k = \frac{1}{10}$.

6. Outlook

Similar perturbative methods could be applied, for example to the case of the Coulomb potential. This should allow us to obtain the relation between the scale of an assumed fundamental non-pointlikeness Δx_0 of the electron and the scale of the thereby caused effects on the hydrogen spectrum. The results of high precision hydrogen spectroscopy are very closely matched by perturbative field theoretical calculations which, by involving higher-order perturbation theory, probe high energies and small distances. Since spectroscopy is low energy, the scale of the sensitivity of the spectroscopic data on an assumed fundamental non-pointlikeness of the electron in the form of a nonvanishing Δx_0 should be of interest, even if the so obtained upper bound for Δx_0 will not be as small as the well known present accelerator-based experimental upper bound of about 1 TeV or 10^{-16} cm.

In the context of a possible fundamental Δx_0, recent studies on minimal uncertainties and regularization in field theory are [9, 20], with related studies being [23, 24].

References

[1] Townsend P K 1976 *Phys. Rev.* D **15** 2795
[2] Gross D J and Mende P F 1988 *Nucl. Phys.* B **303** 407
[3] Amati D, Cialfaloni M and Veneziano G 1989 *Phys. Lett.* B **216** 41
[4] Maggiore M 1993 *Phys. Lett.* B **319** 83
[5] Lizzi F and Mavromatos N E 1996 *Preprint* OUTP-96-66-P, DSF 53/96, hep-th/9611040.
[6] Garay L J 1995 *Int. J. Mod. Phys.* A **10** 145
[7] Witten E 1996 *Physics Today* **49** 24
[8] Kempf A 1994 *J. Math. Phys.* **35** 4483
[9] Kempf A 1996 *Phys. Rev.* D **54** 5174
[10] Kempf A, Mangano G and Mann R B 1995 *Phys. Rev.* D **52** 1108
[11] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
[12] Fiore G 1992 *Int. J. Mod. Phys.* A **7** 7597

[13] Majid S 1995 *Foundations of Quantum Group Theory* (Cambridge: Cambridge University Press)
[14] Carow-Watamura U and Watamura S 1995 *Prog. Theor. Phys. Suppl.* **118** 375
[15] Chung W S and Klimyk A U 1996 *J. Math. Phys.* **37** 917
[16] Meljanac S and Milekovic M 1996 *Int. J. Mod. Phys.* A **11** 1391
[17] Connes A 1994 *Noncommutative Geometry* (New York: Academic)
[18] Kempf A 1992 *Lett. Math. Phys.* **26** 1–12
[19] Kempf A 1997 *Quantum Groups and Quantum Spaces* ed R Budzynski, W Pusz and S Zakrzewskei (Warsaw: Banoch Center Publications)
[20] Kempf A 1996 *J. Math. Phys.* DAMTP/96-22, hep-th/9602085 in press
[21] Hinrichsen H and Kempf A 1996 *J. Math. Phys.* **37** 2121
[22] Kempf A and Mangano G 1996 *Preprint* DAMTP/96-101, hep-th/9612082
[23] Doplicher S, Fredenhagen K and Roberts J E 1994 *Phys. Lett.* B **331** 39
[24] Grosse H, Klimcik C and Presnadjer P 1996 *to appear in Proc. Int. Workshop on Lie Theor. Appl. Phys., Clausthal, Aug 95, W.Sc.* hep-th/9510177